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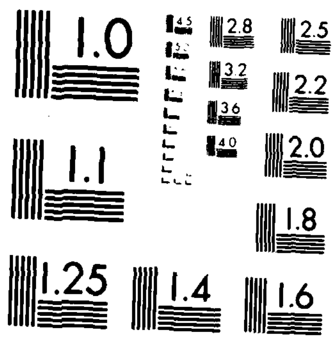

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AN ITERATIVE SCHEME FOR LINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT

An iterative scheme is given for solving the linear complementarity problem $x \geqq 0$, $Mx + q \geqq 0$, $x^T(Mx + q) = 0$. At each iteration the main effort is to find a point x satisfying $x \geqq 0$, $Mx + q \geqq 0$ and a (scalar) linear inequality which is easily updated iteratively. It is shown that if M is a P-matrix or positive semidefinite, or satisfies a fairly weak positivity assumption, then the scheme converges to a solution. The scheme allows many possible implementations including *sparsity-preserving* methods and direct pivoting methods. An iterative linear programming method, LCP-ILP, is discussed in detail. The method is very similar to the Two-Phase method of linear programming. Numerical results are presented which indicate that LCP-ILP is more effective than Lemke's method in solving general linear complementarity problems.

AMS (MOS) Classification: 65K05, 90C33

Key Words: Convergence, iterative methods, linear complementarity, pivoting, simplex method

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SIGNIFICANCE AND EXPLANATION

The linear complementarity problem (LCP) is an important problem in mathematical programming for solving linear programming, quadratic programming and bimatrix games. This work gives a new scheme for solving the LCP. Based on the scheme different algorithms may be designed for finding approximate solutions of very large scale problems or exact solutions of moderate LCPs. A specific algorithm for the latter is given, together with some numerical results. These indicate that the method is more effective than other existing methods.

AN ITERATIVE SCHEME FOR LINEAR COMPLEMENTARITY PROBLEMS

Tzong-Huei Shiau

1. INTRODUCTION

We propose an iterative scheme for solving the linear complementarity problem (LCP) of finding a vector x in the n-dimensional Euclidean space $\mathbb{R}^n$ such that

(1.1) $$x \geq 0, \quad Mx + q \geq 0, \quad x^T(Mx + q) = 0$$

where M is a given n-by-n real matrix and q is a given real n-vector.

At each iteration of the scheme, the main effort is to solve the following system of $2n + 1$ linear inequalities, ($x \geq 0$ is considered n inequalities).

(1.2) $$x \geq 0, \quad Mx + q \geq 0, \quad c_k^T x \leq \alpha_k ,$$

where $c_k \in \mathbb{R}^n$, $\alpha_k \in R$ are easily determined at the k-th iteration. Note that only the last inequality of (1.2) is changed for each iteration. Hence, in solving (1.2), we can start at the solution of the previous iteration which satisfies all except possibly the last inequality. This changing inequality may be considered as a cutting plane which cuts off some protion of the <u>feasible region</u> $\{x | x \geq 0, Mx + q \geq 0\}$. However, the scheme differs from general cutting plane methods in that the number of cuts remains one for each iteration while other cutting plane methods accumulate all the cuts of previous iterations.

It is shown that the scheme generates a "descending" sequence of feasible points of the quadratic program,

(1.3) $$\min f(x) := x^T(Mx + q) \quad \text{subject to} \quad x \geq 0, \quad Mx + q \geq 0$$

and the sequence converges to a Karush-Kuhn-Tucker (KKT) point of (1.3). Note that any global minimum of (1.3) is a solution of (1.1) provided (1.1) has a solution. Therefore in cases where any KKT point of (1.3) is a global minimum, the sequence converges to a solution of (1.1), e.g. when M is positive semi-definite or when M is a P-matrix. The convergence to a solution is finite if in solving (1.2) for a <u>feasible</u> point satisfying the cut, we require that the feasible point is a <u>vertex</u> of the feasible region.

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The scheme is very flexible that it allows many possible implementations. Among them the iterative linear programming method (LCP-ILP) is discussed in detail and some numerical results are presented which indicate that for general linear complementarity problems LCP-ILP is more efficient and solves more porblems than Lemke's method [5]. In some other paper we shall discuss other implementations which preserve sparsity and are more suitable for very large scale problems with sparse M. Unlike most of other sparsity-preserving methods [e.g. see 3, 6, 8, 9], the algorithms can solve nonsymmetric LCP's when M is only positive semidefinite.

We define some notations and terms used in the paper. An $x \in R^n$ is feasible if $x \geqq 0$, $Mx + q \geqq 0$. The set of all feasible points is called the feasible region. The LCP (1.1) is feasible if the feasible region is nonempty. $\nabla f(x) := (M + M^T)x + q$ is the gradient of $f(x) := x^T(Mx + q)$ as defined in (1.3). x is a solution if x is feasible and $f(x) = 0$. x is a vertex if it is an extreme point of the feasible region. By Condition (A), we mean

Condition (A) There exists a solution $\bar{x}$ of (1.1) such that

$$(x - \bar{x})^T M(x - \bar{x}) \geqq 0 \quad \text{for all feasible } x.$$

If (1.1) is feasible, then Condition (A) holds when M is positive semidefinite. In fact, Condition (A) is equivalent to that f, restricted to the feasible region, is convex at some solution $\bar{x}$. From now on we assume (1.1) is feasible.

Superscripts are used to denote different vectors, e.g. x^1, x^k and x^{k+1}, but the superscript T denotes transpose, e.g. x^T, M^T.

2. THE BASIC IDEA

The following lemma shows that given a feasible point which is not a solution, we have a cutting plane separating the point from a solution under a very weak convexity assumption.

2.1 Lemma. Let x^k be feasible and $f(x^k) > 0$. Then the linear inequality

$$f(x^k) + \nabla f(x^k)(x - x^k) \leq 0 \tag{2.1}$$

which is violated by x^k is satisfied by any solution z satisfying

(2.2) $$(x^k - z)^T M(x^k - z) \geq 0 .$$

Proof. Obviously x^k violates (2.1) since $f(x^k) > 0$. Let z be any solution satisfying (2.2). Then the equation

(2.3) $$f(x^k) + \nabla f(x^k)^T(z - x^k) = -(x^k - z)^T M(x^k - z)$$

can be shown either by algebraic manipulation or by equating the zeroth, first and second order derivatives of both sides with respect to x^k at z (note that $f(z) = 0$). It follows by (2.3) and (2.2) that

$$f(x^k) + \nabla f(x^k)^T(z - x^k) \leq 0$$

i.e. z satisfies (2.1). □

Lemma 2.1 leads naturally to the following algorithm:

2.2 The Cutting Plane Method

(0) Initialization. Let S be the set of the $2n$ constraints $x \geq 0$, $Mx + q \geq 0$. Let x^0 be any feasible point.

(1) Iteration. Given x^k, $k \geq 0$, add the cut (2.1) to S, find x^{k+1} satisfying all constraints in S. □

Note that the number of constraints in S increases by one for each iteration. It is simple to establish the following:

2.3 Convergence Theorem of the Cutting Plane Method. Let $\{x^k\}$ be generated by Method 2.2. Then any accumulation point z of $\{x^k\}$ is a solution.

Proof. Let $\{x^{k_j}\}$ be a subsequence converging to z. Since $x^{k_{j+1}}$ satisfies all previous cuts, in particular it satisfies the cut defined by x^{k_j}. Hence

$$f(x^{k_j}) + \nabla f(x^{k_j})^T(x^{k_{j+1}} - x^{k_j}) \leq 0$$

Taking the limit as $j \to \infty$, we have $f(z) \leq 0$. Hence z is a solution since it is clearly feasible. □

It should be pointed out that Theorem 2.3 did not say that the set of points satisfying all constraints in S is always nonempty as more and more cuts are added into S. However, it is indeed the case when Condition (A) holds since $\bar{x}$ satisfies all cuts by Lemma 2.1.

Like other cutting plane algorithms, the method suffers from the following difficulties. As the number of constraints increases, x^{k+1} becomes more and more difficult to compute, and usually the convergence becomes slower and slower. Forturnately, we have found a simple remedy. By doing an additional line search in each iteration, none of previous cuts need be kept.

3. THE ITERATIVE SCHEME

3.1 The Basic Scheme

(0) Initialization. Let x^0 be any feasible point.

(1) Iteration. Given x^k, stop if $f(x^k) = 0$. Otherwise let y^k be any feasible point satisfying the cut (2.1). Line-search along the direction $p^k := y^k - x^k$ using the minimization stepsize $x^{k+1} := x^k + t_k p^k$ where

$$t_k := \arg \underset{0 \leq t \leq 1}{\text{minimum}} \ f(x^k + tp^k) \ . \qquad \square \tag{3.1}$$

3.2 Theorem. Let $\{x^k, y^k\}$ be generated by the *Iterative Scheme* 3.1. *If* $\{y^k\}$ is bounded, then

$$f(x^k) \downarrow 0 \quad \text{as} \quad k \to \infty \ .$$

Proof. See Appendix. $\square$

It should be noted again that without any convexity assumption such as Condition (A), there may be no feasible point satisfying (2.1) in the k-th iteration. We call (2.1) an overcut if it is the case. The following lemma shows that we can still get a descent direction p^k by a weaker cut parallel to the overcut.

3.3 Lemma. Let x^k be feasible. Then

$$\nabla f(x^k)^T(x - x^k) \geqq 0 \quad \text{for all feasible} \quad x \tag{3.2}$$

iff

$$x^k \text{ is a KKT point of } (1.3) \ . \tag{3.3}$$

Therefore, if x^k is not a KKT point of (1.3), then for some $\theta_k > 0$ sufficiently small, there is a feasible point x satisfying

(3.4) $$\nabla f(x^k)(x - x^k) + \theta_k f(x^k) \leq 0 \ .$$

Proof. (3.2) holds iff x^k is an optimal solution of the linear program

(3.5) $$\text{minimize } \nabla f(x^k)x \ \text{ subject to } \ x \geq 0, \ Mx + q \geq 0$$

Since the optimality condition of (3.5) for x^k is precisely the KKT condition of (1.3) for x^k, (3.2) and (3.3) are equivalent. Hence $\nabla f(x^k)(x - x^k) < 0$ for some feasible x if x^k is not a KKT point of (1.3). Therefore (3.4) holds for θ_k sufficiently small. □

The cut (3.4) is a generalization of (2.1) that can be used when (2.1) is an overcut. This leads the general scheme.

3.4 The General Iterative Scheme

(0) Initialization. Let x^0 be any feasible point

(1) Iteration. Given x^k, stop if x^k is a KKT point of (1.3). Otherwise, let y^k be a feasible point satisfying

(3.6) $$\nabla f(x^k)^T(y^k - x^k) + \theta_k f(x^k) \leq 0$$

where θ_k is a positive number. Let $x^{k+1} := x^k + t_k p^k$ where $p^k := y^k - x^k$ and t_k is the minimization stepsize defined in (3.1). □

Ideally θ_k should be 1 if admissible. If $\theta_k = 1$ leads to an overcut, we can keep on reducing it by half until (3.6) holds for some y^k. That is, we choose $\theta_k = 1$ or choose θ_k large enough that $2\theta_k$ will lead to an overcut. The following theorem shows that we will get a solution or a KKT point of (1.3).

3.5 Theorem. Let $\{x^k, y^k\}$ be generated by the General Iterative Scheme 3.4. Assume x^k is not a KKT point of (1.3) for $k \geq 0$. Assume $\{y^k\}$ is bounded. Then

(i) $\{x^k\}$ is bounded, $\{f(x^k)\}$ is decreasing and

$$\lim_{k \to \infty} \nabla f(x^k)^T p^k = 0$$

(ii) If there exists $\delta > 0$ such that for infinitely many k, $\theta_k > \delta$, then

$$\lim_{k \to \infty} f(x^k) = 0$$

(iii) If there exists $\delta > 0$ such that for all k, either $\theta_k \geq \delta$ or

(3.7) $$\nabla f(x^k)^T(x - x^k) + 2\theta_k f(x^k) \leq 0$$

is an overcut, then any accumulation point of $\{x^k\}$ is a KKT point of (1.3).

Proof. See Appendix. □

Theorem 3.5 shows that without any convexity assumption, the scheme will "converge" to a KKT point of (1.3). In cases that we can keep θ_k from *converging to zero, e.g.* $\theta_k = 1$ if Condition (A) holds, then a solution is obtained. The scheme also converges to a solution in cases in which any KKT point is a solution as shown below.

3.6 Corollary. If (i) M is positive semi-definite, or (ii) M is a P-matrix, or (iii) M is quasi-diagonally dominant, i.e. there exists $d = (d_1, d_2, \ldots, d_n)^T \in R^n$, $d_i > 0$, such that

$$M_{ii} d_i \geq \sum_{j \neq i} |M_{ij}| d_j, \quad i = 1,2,\ldots,n, \tag{3.8}$$

then any accumulation point of $\{x^k\}$ is a solution.

Proof. For (i), $f(x)$ is convex. Hence any KKT point of (1.3) is a global minimum of (1.3) which is a solution of (1.1). For (ii), any KKT point is a solution [1], which is also true for (iii) [10, Thm. 5.5.1]. □

4. A SIMPLEX-BASED IMPLEMENTATION

The main effort in each iteration is to find a feasible y^k satisfying the cut. The quality of y^k will affect the speed of convergence of the method. In this section we present a simplex-based method to find y^k which is a vertex of the feasible region. This is desirable because at least one of the vertices is a solution if (1.1) is solvable [7].

4.1 The Simplex-Based Method (LCP-ILP)

(0) Initialization. Use phase I of the two-phase simplex method to find a basic feasible solution (BFS) and the corresponding simplex tableau for the linear program

$$\min 0 \cdot x \quad \text{subject to} \quad Mx + q \geq 0, \quad x \geq 0 . \tag{4.1}$$

Let $x^1 = y^0$ = the BFS. (This step is not needed if a BFS is available).

(1) Iteration. Given x^k feasible and a vertex y^{k-1} and its tableau, recompute the reduced cost vector using $\nabla f(x^k)$ as the new cost vector, then continue the primal pivot-steps until either (i) the current BFS satisfies the cut (2.1), or

(ii) the BFS is optimal for the cost vector $\nabla f(x^k)$, whichever occurs first.

Let y^k be the BFS, compute x^{k+1} by the line search described before. □

Roughly speaking, the method is very close to the Two-Phase Simplex method for solving the linear program

$$\min c^T x \quad \text{subject to} \quad Mx + q \geq 0, \quad x \geq 0 \; . \tag{4.2}$$

The only difference is that $c = \nabla f(x^k)$ is updated whenever the current BFS y^k satisfies the cut (2.1), i.e. the objective

$$c^T y^k = \nabla f(x^k)^T y^k < \nabla f(x^k)^T x^k - f(x^k) = x^k M^T x^j =: \alpha_k \tag{4.3}$$

which is very easy to test, or when y^k is optimal for $c = \nabla f(x^k)$. In the latter case, y^k satisfies the generalized cut (3.4) with

$$\theta_k := - \frac{\nabla f(x^k)^T (y^k - x^k)}{f(x^k)} \geq 0 \tag{4.4}$$

where $\theta_k = 0$ iff x^k is optimal for (4.2) (with $c = \nabla f(x^k)$). If $\theta_k = 0$ then x^k is a KKT point of (1.3), otherwise x^{k+1} is determined by a simple line-search and c and α_k (defined in (4.3)) are easily updated. Note that $2\theta_k$ leads to an overcut since θ_k is the largest number which does not.

If the reader is careful enough, she/he may ask what if the simplex method leads to a unbounded ray. This is the last problem we need to solve before the convergence results in Section 3 are applicable. The answer is very simple: it never happens.

4.2 Theorem. Let x^k be feasible, then the LP

$$\min \nabla f(x^k)^T y \quad \text{subject to} \quad My + q \geq 0, \quad y \geq 0$$

is bounded.

Proof. Let d be any direction of the feasible region, i.e. there is a feasible y such that $y + \lambda d$ is feasible for all $\lambda \geq 0$. Then $d \geq 0$ and $Md \geq 0$. Hence

$$\nabla f(x^k)^T d = (Mx^k + q)^T d + (x^k)^T (Md) \geq 0 \; .$$

Therefore no direction is a descent direction and LP is bounded. □

The following theorem shows that the convergence of Simplex-Based Method (LCP-ILP) is finite if it is not trapped by a non-solution KKT point. Hence if θ_k is away from zero,

e.g. Condition (A) holds, or one of the conditions of Corollary 3.5 is true, then $x^{k+1} = y^k$ is a solution for some k.

4.3 Theorem. Let $U := \{z | z$ is a vertex, but not a solution$\}$ be the finite set of non-solution vertices. Let $\mathrm{conv}(U)$ be its convex hull, let $\delta := \inf\{f(x) | x \in \mathrm{conv}(U)\}$. Then LCP-ILP either

(i) generates a sequence $\{x^k\}$ such that $f(x^k) \geq \delta > 0$ for all k, or (ii) terminates at a solution x^k for some $k \geq 1$, in which case y^{k-1} is a vertex solution.

Proof. Since U is finite, $\mathrm{conv}(U)$ is compact. Hence there is a vector, say x, in $\mathrm{Conv}(U)$ such that

$$f(x) = \delta = \inf\{f(z) | z \in \mathrm{conv}(U)\} .$$

Let $x = \Sigma\lambda_k z^k$, $z^k \in U$, $\lambda_k > 0$, $\Sigma\lambda_k = 1$. Picking any z^k, say z^1, there is an index i, $1 \leq i \leq n$, such that $z_i^1 > 0$ and $(Mz^1 + q)_i > 0$ since z^1 is not complementary. Hence

$$x_i = \Sigma\lambda_k z_i^k \geq \lambda_1 z_i^1 > 0 ,$$

and

$$(Mx + q)_i = \Sigma\lambda_k (Mz^k + q)_i \geq \lambda_1 (M_z^1 + q)_i > 0 .$$

So

$$\delta = f(x) = \sum_{j=1}^{n} x_j (Mx + q)_j \geq x_i (Mx + q)_i > 0 .$$

Suppose $y^k \in U$ for $0 \leq k \leq N$, then it is easy to see, by induction, that $x^{N+1} \in \mathrm{conv}(U)$. So $f(x^{N+1}) \geq \delta > 0$. This shows that either one of y^k is a vertex solution or $f(x^k) \geq \delta$ for any x^k generated. In the formal case, $x^{k+1} = y^k$ is the solution obtained by the line-search. □

4.4 Corollary. LCP-ILP finds a vertex solution in a finite number of iterations if one of the followings is true.

(i) Condition (A) holds.

(ii) M is positive semi-definite.

(iii) M is a P-matrix.

(iv) M is quasi-diagonally dominant.

Proof. For (i), the cut (2.1) is not an overcut. Therefore $\theta_k \geq 1$ at each iteration. It follows by Theorem 3.5 (ii) that case (i) of Theorem 4.3 is impossible.

For (ii), (iii) and (iv), any KKT point is a solution as is pointed out in the proof of Corollary 3.6. Suppose LCP-ILP does not find a solution in a finite number of iterations, we shall show a contradiction. By Theorem 3.5, since x^k, $k \geq 0$, are not KKT points, LCP-ILP generates a bounded sequence $\{x^k\}$ having an accumulation point, say $\bar{x}$, which is a KKT point. Hence $f(\bar{x}) = 0$ since a KKT point is a solution. On the other hand, $f(x^k) \geq \delta > 0$ for all k by Theorem 4.3. Hence $f(\bar{x}) \geq \liminf_{k\to\infty} f(x^k) \geq \delta > 0$. □

4.5 Numerical Results. A computer program implementing LCP-ILP was written and tested on UWVAX, a VAX 11/780 computer running UNIX operating system. All floating point computations are in dobule precision which provides about 16 figure decimal accuracy. Six problem sets were randomly generated; each has 20 individual problems. The dimension n is fixed in each set and varies among sets. For the purpose of comparison Lemke's Algorithm was also tested for these problems. Recall that LCP-ILP may converge to a KKT point which is not a solution, while Lemke's Algorithm may terminate at a ray and fail to give a solution. In fact, since the general LCP is NP-complete [2], there is no known algorithm that is guaranteed to solve any LCP without essentially enumerating an exponential number of possible cases which is impossible to do even for moderate n.

The results are shown in Table 1. We make the following remarks.

(i) The iteration number and the pivot number are the same for Lemke's method, but different for LCP-ILP. For the latter, iteration number is the number of updatings of the cost vector while the pivot number is the total number of pivots of all iterations. Since updating the cost vector is easy and the number of updating is significantly smaller than the pivot number, we feel the pivot number is a good measure of computation time. Also the comparison between pivot numbers of these two algorithms is a good estimate of the relative speed since each pivot takes roughly the same time in either algorithm.

(ii) The phase I of LCP-ILP starts at the origin to find the initial BFS. The number of pivots given in Table 1 includes the pivot steps taken in phase I.

(iii) Although Lemke's Algorithm is very effeicient for M being a P-matrix or positive semi-definite, it rarely gives a solution for a general M when $n \geq 30$. Moreover, it becomes very slow no matter whether it gives a solution or not. Hence the so called almost complementary path has a very long average length.

(iv) The number of problems solved in a set decreases rapidly as n increases. This is not surprising since the problem is NP-complete [2].

(v) The ratio of the maximum number to the minimum number of pivots of a set, given in the bottom row of each column, remains small for LCP-ILP but varies considerably for Lemke's method. Hence the almost complementary path can be very short as well as very long. It also indicates that LCP-ILP is more robust.

(vi) Since LCP-ILP can start at any BFS, it will be much faster if we have a BFS to begin with or each problem set is a parametric LCP in which case it can start at the solution basis of the previous problem. It should be noted, however, that some generalizations of Lemke's method can start at an arbitrary point under certain assumptions [4, 11].

Computational Results of ILP Versus Lemke's Method							
Dimension of Problem ($=n$)		7	15	23	31	40	50
Number of Problems Solved Out Of 20 Problems	ILP	17	11	11	9	7	6
	Lemke	13	7	2	2	1	0
Average Number of Pivots For Solved Problems	ILP	4	15	35	45	90	144
	Lemke	5	21	40	231	205	-
Average Number of Pivots For Unsolved Problems	ILP	2	16	44	55	98	136
	Lemke	7	14	39	102	231	514
Maximum Number of Pivots Over the 20 Problems	ILP	8	29	75	101	129	234
	Lemke	17	47	91	493	1000*	1000*
Minimum Number of Pivots Over the 20 Problems	ILP	1	0	16	29	73	85
	Lemke	2	0	8	1	2	11
Ratio Of Max/Min of Pivots	ILP	8.0	-	4.7	3.5	1.8	2.8
	Lemke	8.5	-	11.4	493	500*	90.9*

* (quit after 1000 iterations, so the true value should be bigger)

Table 1

APPENDIX

Theorem 3.2 is a special case of Theorem 3.5 with $\theta_k = 1$. (Note that x^k is not a KKT point in Theorem 3.2. Otherwise by Lemma 3.3 there would be no point satisfying the cut (2.1) and x^{k+1} could not be generated). Hence we shall only give the proof of Theorem 3.5.

For simplicity, we introduce the following notation.

(A.1) $$\phi(t) := f(x^k + tp^k), \quad t \in R$$

(A.2) $$\alpha := f(x^k), \quad \beta := \nabla f(x^k)^T p^k, \quad \gamma := (p^k)^T M p^k$$

It is easy to see that $\phi(t)$ is a quadratic function and that $\phi(0) = \alpha$, $\phi'(0) = \beta$, $\phi'(0) = \partial\gamma$. Hence

(A.3) $$\phi(t) \equiv \alpha + \beta t + \gamma t^2$$

<u>Lemma A.1</u>. Let $\bar{t} := \frac{-\beta}{2\gamma}$, then

(i) $t_k = \begin{cases} \bar{t} & \text{if} \quad 0 < \bar{t} < 1 \\ 1 & \text{otherwise} \end{cases}$

where t_k is the stepsize defined by (3.1).

(ii) $f(x^k) - f(x^{k+1}) \geqq -\frac{1}{2} t_k \beta > 0$

<u>Proof</u>. By assumption x^k is not a KKT point, hence $\alpha > 0$. By (3.6), $\beta \leq -\theta_k \alpha < 0$ since θ_k is positive.

<u>Case 1</u>. $0 < \bar{t} < 1$.

Hence $\gamma > -\frac{1}{2}\beta > 0$, $\phi(t)$ is strictly convex with the global minimum $\bar{t}$. Hence $t_k = \bar{t}$ and

$$f(x^{k+1}) - f(x^k) = t_k(\beta + t_k\gamma) = t_k\left(\beta + \frac{-\beta}{2\gamma}\gamma\right) = \frac{1}{2} t_k\beta$$

proving (ii).

<u>Case 2</u>. $\bar{t} \geq 1$.

So $0 < \gamma \leq -\frac{1}{2}\beta$ and $\phi(t)$ is again strictly convex. Since $\bar{t} \geq 1$, $\phi(t)$ is strictly decreasing on [0,1]. So $t_k = 1$ and

$$f(x^{k+1}) - f(x^k) = \beta + \gamma \leq \beta - \frac{1}{2}\beta = \frac{1}{2} t_k\beta$$

proving (ii).

Case 3. $\bar{t} \leqq 0$.

So $\gamma \leqq 0$ and $\phi(t)$ is strictly decreasing on $[0,1]$. Hence $t_k = 1$ and

$$f(x^{k+1}) - f(x^k) = \beta + \gamma \leqq \beta < \frac{1}{2} t_k \beta .$$

The proof is complete. □

Proof of Theorem 3.5. Since $\{y^k\}$ is bounded, there is a positive number V, that

$$|x^0| < V, \quad |y^k| < V \text{ for all } k \geqq 0 .$$

It is easy to see, by induction, that x^{k+1} is a convex combination of $x^0, y^0, \ldots, y^k$ and hence

$$|x^k| < V \text{ for all } k \geqq 0 .$$

Hence $|p^k| = |y^k - x^k| < |y^k| + |x^k| < 2V$ and

$$|\gamma| = |(p^k)^T M p^k| < \|M\| \cdot |p^k|^2 < \|M\| \cdot 4V^2 =: W \tag{A.4}$$

for all $k \geqq 0$. Now applying Lemma A.1 (i) and (ii), we have

$$\begin{aligned} f(x^k) - f(x^{k+1}) &\geqq -\frac{1}{2} t_k \beta \\ &\geqq \min\left\{-\frac{1}{2}\beta, -\frac{\beta}{2}\frac{-\beta}{2|\gamma|}\right\} \\ &= \min\left\{-\frac{1}{2}\beta, \frac{\beta^2}{4|\gamma|}\right\} \\ &> \min\left\{-\frac{1}{2}\beta, \frac{\beta^2}{4W}\right\} =: \delta_k \qquad \text{(by (A.4))} \end{aligned} \tag{A.5}$$

Summing up (A.5) for $k = 0, \ldots, N$, we have

$$f(x^0) - f(x^{N+1}) > \sum_{k=0}^{N} \delta_k \tag{A.6}$$

The positive series $\sum_{k=0}^{\infty} \delta_k$ converges since the left hand side of (A.6) is bounded for all N. Hence $\lim_{k\to\infty} \delta_k = 0$. It follows by the definition of δ_k in (A.5) that

$$\lim_{k\to\infty} -\beta = \lim_{k\to\infty} -\nabla f(x^k)^T p^k = 0$$

So by (3.6),

$$\text{(A.7)} \qquad \limsup_{k\to\infty} \theta_k f(x^k) \leqq \lim_{k\to\infty} -\nabla f(x^k)^T p^k = 0$$

On the other hand, since $\{f(x^k)\}$ is strictly decreasing by (A.5), if $\theta_k > \delta$ infinitely often we have

$$\limsup_{k\to\infty} \theta_k f(x^k) > \left(\limsup_{k\to\infty} \theta_k\right)\cdot\left(\lim_{f\to\infty} f(x^k)\right) > \delta\cdot\lim_{k\to\infty} f(x^k)$$

Together with (A.7), we have $\lim\limits_{k\to\infty} f(x^k) = 0$ proving (ii).

For (iii), because of (ii) we can assume that (3.7) is an overcut for all $k > K$ for some K. Fix any feasible y. For $k > K$, since y cannot satisfy (3.7), we have

$$\nabla f(x^k)^T(y - x^k) > -2\theta_k f(x^k)$$

so

$$\liminf_{k\to\infty} \nabla f(x^k)^T(y - x^k) \geqq \liminf_{k\to\infty} -2\theta_k f(x^k)$$

$$= -2 \limsup_{k\to\infty} \theta_k f(x^k) > 0 \qquad \text{(by (A.7))}$$

Let $\bar{x}$ be any accumulation point of $\{x^k\}$, then $\nabla f(\bar{x})^T(y - \bar{x})$ is an accumulation point of $\{\nabla f(x^k)^T(y - x^k)\}$, hence

$$\text{(A.8)} \qquad \nabla f(\bar{x})^T(y - \bar{x}) \geqq \liminf_{k\to\infty} \nabla f(x^k)^T(y - x^k) > 0$$

Since (A.8) holds for any feasible y, $\bar{x}$ is a KKT point of (1.3) by Lemma 3.3. □

REFERENCES

1. Y. C. Cheng, "On the gradient projection method for non-symmetric linear complementarity problems", JOTA Aug. 1984.
2. S. J. Chung and K. G. Murty, "Polynomially bounded ellipsoid algorithms for convex quadratic programming", Nonlinear Programming 4, ed. O. L. Mangasarian, R. R. Meyer and S. M. Robinson, Academic Press, 1981, 439-485.
3. C. W. Cryer, "The solution of a quadratic programming problem using systematic overrelaxation", SIAM Journal of Control 9, 1971, 385-392.
4. B. C. Eaves, "Computing stationary points", Mathematical Programming Study 7, 1978, 1-14.
5. C. E. Lemke, "On complementary pivot theory", Mathematics of the Decision Sciences, ed. G. B. Dantzig and A. F. Veinott, Jr., American Math. Soc., 1968.
6. O. L. Mangasarian, "Solution of symmetric linear complementarity problems by iterative methods", Journal of Optimization Theory and Applications 22, 1977, 465-485.
7. O. L. Mangasarian, "Characterization of linear complementarity problems as linear programs", Mathematical Programming Study 7, 1978, 74-87.
8. J. S. Pang, "On the convergence of a basic iterative method for the implicit complementarity problem", Journal of Optimization Theory and Applications 17, 1982, 149-162.
9. J. S. Pang, "Necessary and sufficient conditions for the convergence of iterative methods for the linear complementarity problem", to appear in Journal of OPtimization Theory and Applications.
10. T.-H. Shiau, "Iterative linear programming for linear complementarity and related problems", Technical Report #507, Department of Computer Sciences, University of Wisconsin, 1983.
11. D. Talman and L. van der Heyden, "Algorithms for the linear complementarity problem which allow an arbitrary starting point", Cowles Foundation Discussion Paper No. 600, Cowles Foundation for Research in Economics, Yale University, 1981.

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An iterative scheme is given for solving the linear complementarity problem $x \geqq 0$, $Mx + q \geqq 0$, $x^T(Mx + q) = 0$. At each iteration the main effort is to find a point x satisfying $x \geqq 0$, $Mx + q \geqq 0$ and a (scalar) linear inequality which is easily updated iteratively. It is shown that if M is a P-maxtix or positive semidefinite, or satisfies a fairly weak

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ABSTRACT (cont.)

positivity assumption, then the scheme converges to a solution. The scheme allows many possible implementations including sparsity-preserving methods and direct pivoting methods. An iterative linear programming method, LCP-ILP, is discussed in detail. The method is very similar to the Two-Phase method of linear programming. Numerical results are presented which indicate that LCP-ILP is more effective than Lemke's method in solving general linear complementarity problems.

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